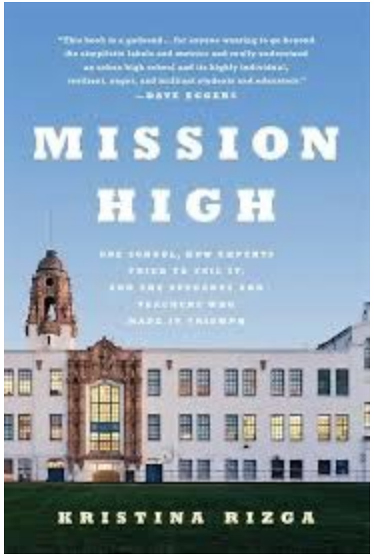
Conceptual and Procedural
EWA Conference



Kentaro Iwasaki

https://www.concentricmath.com/

CONCENTRIC MATH

The Game of Pig

Goal: Get the most points.

How to Play: Each turn of the game consists of one or more rolls of the die. You keep rolling until you decide to stop or until you roll 1. You may choose to stop rolling at any time.

Scoring

- If you choose to stop rolling before you roll 1, your score for that turn is the sum of all the numbers you rolled on that turn.
- However, if you roll 1, your turn is over, and your score for that turn is 0.

Examples

• You roll 4, 5, and 2, and then decide to stop. Your score for this turn is 11.

• You roll 3, 4, 6, and 1. The turn is over because you rolled 1, and your score for this turn is 0.

Each turn is scored separately. With each turn you play, you add the score for that turn to the total of your previous turns. Record points for each turn.

An Example of Teaching for Conceptual Understanding

1. Start with a large question that is too hard for students to solve immediately based on their current knowledge (such as the best strategy for “Pig”) that anchors students for 4-6 weeks
   a. They will seek to answer this question and seek resolution
2. Have students engage in daily lessons in which they work to learn the math needed for the large question in a sequential manner through problem solving
   a. Make these daily lessons as engaging and relevant as possible
3. Incorporate skills practice and procedures based on the math needed each day with review and preview as needed
4. Formalize math formulas explicitly with students once they understand where the formulas come from so they can use formulas and shortcuts as needed
5. Spiral skills/procedures practice with the daily problem solving
6. Assess students on the skills/procedures as well as the concepts and applications of the mathematics
7. INTERACTIVE MATHEMATICS PROGRAM (IMP)

One-and-One

Shaquille O'Neal (Shaq) is a 60% free throw shooter.

In a one-and-one situation, how many points is Shaq most likely to score?

In a one-and-one situation, the player begins by taking a free throw. If the player misses, that's the end of it. But if the shot is successful, the player gets to take a second shot.

One point is scored for each successful shot. So the player can end up with 0 points (by missing the first shot), 1 point (by making the first shot, but then missing the second), or 2 points (by making both the first and the second shots).

The Interactive Mathematics Program https://activatelearning.com/high-school-curriculum/interactive-mathematics-program-imp/

The Birthday Problem(s)

How likely is it that two people in this room have the same birthday? What's the probability?

The Real Birthday Problem: What is the minimum number of people you need so that the probability of having at least one birthday match is greater than ½ ?

CONCENTRIC MATH

The Interactive Mathematics Program https://activatelearning.com/high-school-curriculum/interactive-mathematics-program-imp/

Let’s Make a Deal!

There was a famous game show called “Let’s Make a Deal.” In it, you are shown three doors, labeled A, B,and C. Behind one of the doors is a brand new car of your dreams. Behind the other two doors are worthless prizes. You select a door. The host of the game then opens one of the other doors. Because the host knows where the car is, he makes sure to open a door that reveals a worthless prize. You now have to make a choice: You can either stay with the door you originally selected or switch to the remaining closed door. You win whatever is behind the door you choose at this time.

Are your chances of winning the car better if you stay with your original choice, or are they better if you switch to the remaining closed door, or are they equally likely with the two strategies? More precisely, your task is to find the probability of getting the car for each of the two strategies. Write a careful explanation of how you found these probabilities.

The Interactive Mathematics Program https://activatelearning.com/high-school-curriculum/interactive-mathematics-program-imp/

Is Math a ladder… …or a web?

Or math as a webbed ladder?

Math as a three-legged stool

-Phil Daro, writer of the Common Core Math Standards

Skills/Procedures

Application to the Real World

Conceptual Understanding

Some definitions of conceptual and procedural knowledge

Conceptual Knowledge	Procedural Knowledge
● Connected web of knowledge ● Network in which the linking relationships are as prominent as discrete pieces of information	● Familiarity with symbols, syntax, and conventions ● Rules and procedures (chains of prescriptions)

- *Conceptual knowledge* is typically defined as:
  ... knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. (Hiebert & Lefevre, 1986, pp. 3-4).
- *Procedural knowledge* is defined in terms of two kinds of knowledge:
  One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols (Hiebert & Lefevre, 1986, pp. 7-8).

- Rittle-Johnson and Alibali's (1999) widely cited empirical study on conceptual and procedural knowledge in mathematics defined conceptual and procedural knowledge as follows:

  We define *conceptual knowledge* as explicit or implicit understanding of the principles that govern a domain and of the interrelations between pieces of knowledge in a domain. We define *procedural knowledge* as action sequences for solving problems. (p. 175)

- Byrnes and Wasik (1991) noted the following:

  Conceptual knowledge, which consists of the core concepts for a domain and their interrelations (i.e., "knowing that"), has been characterized using several different constructs, including semantic nets, hierarchies, and mental models. Procedural knowledge, on the other hand, is "knowing how" or the knowledge of the steps required to attain various goals. Procedures have been characterized using such constructs as skills, strategies, productions, and interiorized actions. (p. 777)

Humans are **born with an intuitive sense of numerical magnitude** (Feigenson, Dehaene, and Spelke, 2004).

Human minds **want to see and understand patterns** (Devlin, 2006)

In the earliest grades, young students' work in mathematics is **firmly rooted in their experiences** in the world (Piaget and Cook, 1952)

Students learn best when they are actively engaged in questioning, struggling, problem solving, reasoning, communicating, making connections, and explaining—in other words, when they are making sense of the world around them. The research is clear that powerful mathematics classrooms are places that nurture student agency in math. Students are willing to engage in "productive struggle" because they believe their efforts will result in progress. They understand that the intellectual authority of mathematics rests in mathematical reasoning itself—mathematics makes sense! (Nasir, 2002; Gresalfi et al., 2009; Martin, 2009; Boaler and Staples, 2008)

[Mathematics Framework]

Examples of Tasks that emphasize conceptual understanding in math

the Mathematics Assessment Project

Research on the Shell Center Formative Assessment Lessons/Classroom Challenges

In Education Mathematics Matter Video of students in Washington Heights NYC engaged in tasks that require conceptual understanding

Classroom and student experiences with the Interactive Mathematics Program

Rehumanizing Mathematics

Rochelle Gutierrez | Illinois

Rochelle Gutiérrez: "Rehumanizing Mathematics: A Vision for the Future" Video

Rochelle Gutierrez University of Illinois, Urbana-Champaign | UIUC · Department of Curriculum and Instruction

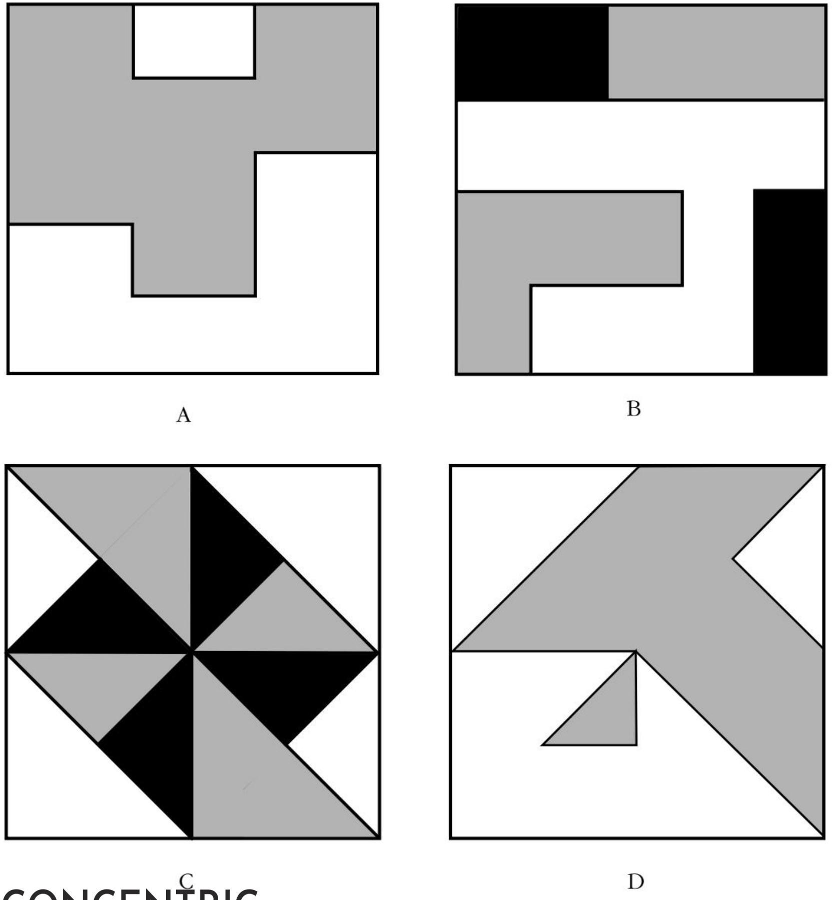
…affirming students' full identities as mathematicians and move them **from merely "playing the game" of school mathematics to "changing the game" by addressing power dynamics** inherent in our education system

Although many concepts can contribute to rehumanizing mathematics for students and teachers who are Latinx, Black, and Indigenous, eight dimensions stand out for me. They include: (1) **participation/positioning**, (2) cultures/histories, (3) windows/mirrors, (4) living practice, (5) creation, (6) broadening mathematics, (7) **body/emotions**, and (8) **ownership**…**connection, joy, and belonging**.

Goffney, I., Gutiérrez, R., & Boston, M. (2018). *Rehumanizing mathematics for Black, Indigenous, and Latinx students (Annual perspectives in mathematics education 2018)*. Reston, VA: National Council of Teachers of Mathematics.

What does practice look like?

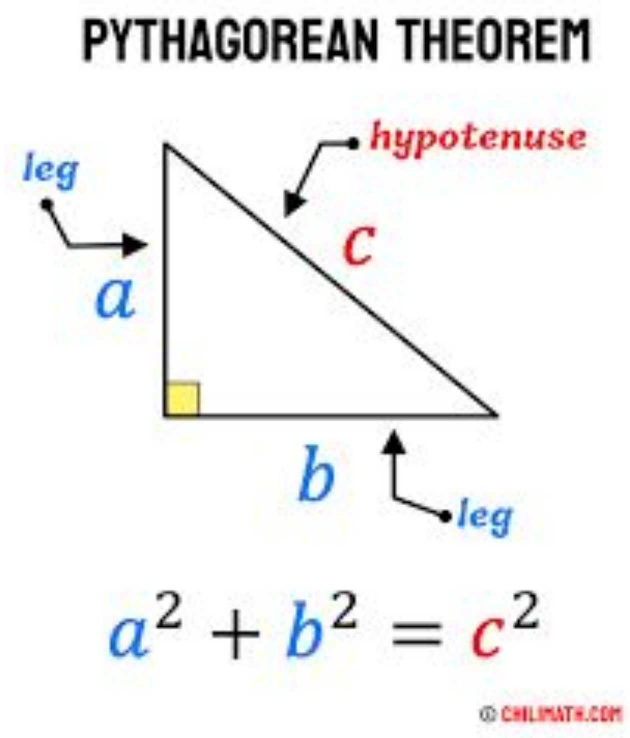
2. For each of the rugs below, decide which color you would predict as most likely to be hit. For each color, find the probability of being hit.



Find the area of the grey, white, and black areas as **fractions, decimals and percents** to link to the probability unit.

conversion_worksheet.docx

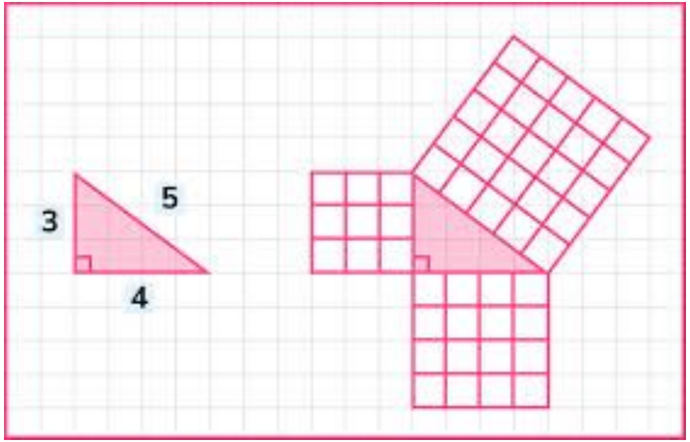
Sophia Argueta-Moran (representative of students all over)

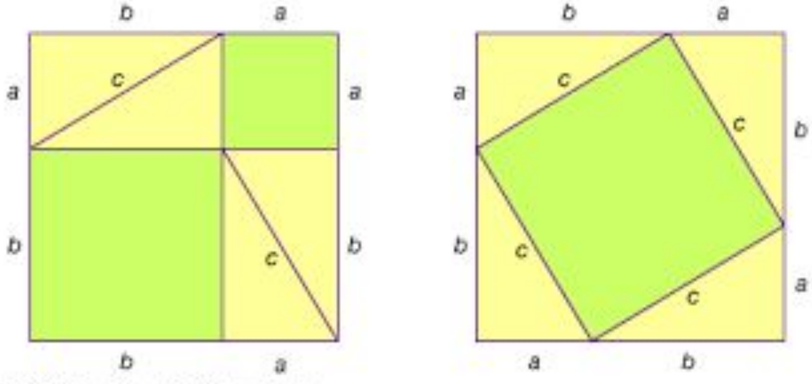


Pythagorean Theorem approaches

Practice problems are easy to find

Addressing misconceptions and correcting them are much harder–need to get to the "why" and the root of the misconception and re-establish a correct math "habit"





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Sophia Argueta-Moran

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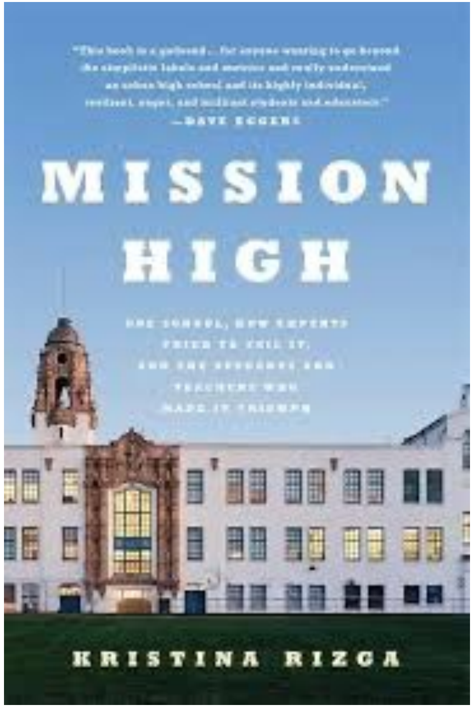

All mathematical ideas can be considered in different ways—visually; through touch or movement; through building, modeling, writing and words; through apps, games and other digital interfaces; or through numbers and algorithms. The tasks used in classrooms should offer multiple ways to engage with and represent mathematical ideas. Multiple representations can help maintain the high cognitive demand of the task for students (Stein et al., 2000) and invite students to engage in the ideas visually; through touch or movement; through building, modeling, writing and words; through apps, games and other digital interfaces; or through numbers and algorithms. Such tasks have been found to support students with learning differences (Lambert and Sugita, 2016) as well as high achievers seeking greater challenges (Freiman, 2018). The guidelines in Universal Design for Learning (or UDL), which are designed to support learning for all, illustrate how to teach in a multidimensional way using multiple forms of engagement, representation, and expression (CAST, 2018).

Using a multidimensional approach to mathematics. **Learning mathematical ideas comes not only through numbers, but also through words, visuals, models, algorithms, tables, and graphs; from moving and touching; and from other representations**. Research in mathematics learning during the last four decades has shown that when students engage with multiple mathematical representations and through different forms of expression, they learn mathematics more deeply and robustly (Elia et al., 2007; Gagatsis and Shiakalli, 2004) and with greater flexibility (Ainsworth et al., 2002; Cheng 2000).

Mathematics Framework

How is conceptual understanding misunderstood or caricatured?

- It’s not real math
- It does not focus on procedures or skills
- It’s fuzzy math or “feel good”/”touchy-feely” math
- It’s not rigorous
- We just let students wallow in their misunderstanding for the whole class
- We don’t provide students with strategies or scaffolds
- It’s a waste of time
- Student interest doesn’t matter
- Paolo Freire model of banking: students are empty accounts that need to be filled with knowledge rather than that students bring a wealth of knowledge already that we can build on

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

(a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

(c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

AP Calculus Exam
6 Free Response Questions questions in 90 minutes comprise half of the exam and are similar to questions that appeared on college exams

Examples of Tasks that emphasize conceptual understanding in math

Pig

https://www.map.mathshell.org/lessons.php

Negative space task page 2.docx

"Relative to typical growth in math from eighth to ninth grade, the effect size for MDC (the Math Design Collaborative) represents an additional 4.6 months of schooling."

Examining the Implementation, Scale Up, and Sustainability of Teacher-Developed Formative Assessment Tools Aligned to the CCSS: A National Study - Research for Action

the Mathematics Assessment Project

My work with 20+ Districts: Los Angeles, San Francisco, Oakland, Sacramento, Fresno, Long Beach, San Bernardino County, Detroit, Burlington VT, Cambridge, Somerville, Needham (MA)

Teaching for conceptual understanding is very challenging

We teachers in general did not learn math with an emphasis on conceptual understanding ourselves. We often teach in the way we learned.

Teachers spend a great deal of time on procedural fluency and skills even when they have curriculum that stresses conceptual understanding. Teachers resort to teaching skills.

With time limits, teachers do not get to the parts of lessons or to tasks that require conceptual understanding when they start with procedures because the skill-building takes so much time.

Decades of neuroscience research have revealed that there is **no single “math area” in the brain**, but rather sets of interconnected brain areas that support mathematical learning and performance (Feigenson, Dehaene, and Spelke, 2004; Hyde, 2011). When students engage in mathematical tasks, they are recruiting both **domain-specific and domain-general brain systems, and the pattern of activation across these systems differs depending on the type of mathematical task the students are performing** (Vogel and De Smedt, 2021; Sokolowski, Hawes, and Ansari, 2023). In addition, growing evidence about “brain plasticity” underscores the fact that the more one uses the brain in particular ways, the more capacity the brain has to think in those ways. One study conducted by neuroscientists in Stanford’s School of Medicine examined the effects of a tutoring intervention with students who had been diagnosed as having mathematical “learning disabilities” and those with no identified difficulties in mathematics (Iuculano et al., 2015). Prior to the intervention, the group of students with identified “learning disabilities” had lower mathematics performance and different brain activation patterns than students who had no identified difficulties in mathematics. After eight weeks of one-on-one tutoring focused on strengthening student understanding of relationships between and within operations, not only did both sets of students demonstrate comparable achievement, but they also exhibited comparable brain activation patterns across multiple functional systems (Iuculano et al., 2015). This study is promising, insofar as it suggests that **well-designed and focused math experiences may support brain plasticity that enables students to access and engage more productively in the content.**

Mathematics Framework

Fluency Without Fear - YouCubed

For about one third of students the onset of timed testing is the beginning of math anxiety (Boaler, 2014). Sian Beilock and her colleagues have studied people's brains through MRI imaging and found that math facts are held in the working memory section of the brain. But **when students are stressed**, such as when they are taking math questions under time pressure, **the working memory becomes blocked and students cannot access math facts they know** (Beilock, 2011; Ramirez, et al, 2013). **Math anxiety has now been recorded in students as young as 5 years old** (Ramirez, et al, 2013). **Widespread anxiety and dislike of mathematics that pervades the US and UK** (Silva & White, 2013; National Numeracy, 2014).

The discussion of conceptual/procedural is linked to the tracking/detracking conversation because lower tracked math classes generally focus on procedures

Conceptual Understanding Premises

- We as humans are innately problem solvers
- Through lived experience, we bring an understanding of math (numbers, patterns)
- We need to build off of the knowledge and experiences we have to deepen our understanding of math

The following slides contain some of the research and references on the foundations of conceptual understanding in math

In this framework, *rigor* refers to an integrated way in which conceptual understanding, strategies for problem-solving and computation, and applications are learned so that each supports the other.[1] Using this definition, conceptual understanding cannot be considered rigorous if it cannot be *used* to analyze a novel situation encountered in a real-world application or within mathematics itself (for new examples and phenomena). Computational speed and accuracy cannot be called rigorous unless it is accompanied by conceptual understanding of the strategy being used, including why it is appropriate in a given situation. And a correct answer to an application problem is not rigorous if the solver cannot explain both the ideas of the model used and the methods of calculation.

[Mathematics Framework]

Designing instruction for rigor. Thus, the challenge posed by the principle of *rigor* is to provide all students with experiences that interweave mathematical concepts, problem-solving (including appropriate computation), and application, such that each supports the other. For instruction that embodies rigor:

- Ensure that abstract formulations *follow* experiences with multiple contexts that call forth similar mathematical models.

- Choose varied mathematical contexts for problem-solving that provide different opportunities for students to use skills, content, and representations for important concepts, so that students can later use those contexts to reason about the mathematical concepts raised. The Drivers of Investigation provide broad reasons to think rigorously in ways that enable students to recognize, value, and internalize linkages between and through topics (Content Connections).

- Ensure that computation serves students' genuine need to know, typically in a problem-solving or application context. In particular, in order for computational algorithms (standard or otherwise) to be understood rigorously, students must be able to connect them to conceptual understanding (via a variety of representations, as appropriate) and be able to use them to solve authentic problems in diverse contexts. An important aspect of this understanding is to recognize the power that algorithms bring to problem solving: knowing only single-digit multiplication and addition facts, it is possible to compute any sum, difference, or product involving whole numbers or finite decimals.

- Choose applications that are authentic for students and enact them in a way that requires students to explain or present solution paths and alternate ideas. Support students in the class to use different skills and content to solve the same problem and facilitate discussions to help students understand why different approaches result in the same answer.

Mathematics Framework

The impact of teaching for conceptual understanding–Ana Torres who was a student at Railside High and later becomes a math teacher

Stories from Jo Boaler's Railside High

Copy Room Conversations: Paying it Forward - Lisa and Ana on Apple Podcasts

What is complex instruction? An instructional approach to mitigate status and power issues that get in the way of student learning in heterogeneous classrooms as they engage in tasks that demand conceptual understanding

Complex Instruction - raising achievement through group worthy tasks | NRICH

A few data points...

Jo Boaler “Railside High” study

	Traditional Math	Railside HS (Complex Instruction)
Year 1 Pre-test Score	22.23	16.00
Year 1 Post-test Score	23.90	22.06
Year 2 Post-test Score	18.34	26.47
Year 3 Post-test Score	19.55	21.44
Seniors taking advanced math course	27%	41%

“...extremely successful at reducing the achievement gap among groups of students belonging to different ethnic groups…”

Student Perceptions and Relationships with Math

“Enjoy math,” “like math,” “enjoy math all or most of the time”

	Traditional	Railside (CI)
Year 2 “enjoy math”	46%	71%
Year 3 enjoy math all or most of the time)	29%	54%
“I like math” strongly agree or agree	54%	74%

Student Perceptions and Relationships with Math

“Students at Railside significantly more interested in math and believing they had significantly more authority and agency”

	Traditional	Railside (CI)
Pursue more math courses	67%	100%
Plan a future with math	5%	39%

[A Close Examination of Jo Boaler's Railside Report]

I recognize the pushback on Jo Boaler's Railside High study.

That said, the high level and quality of student engagement and discourse around math were unmatched.

Three classroom videos (some dated) that show levels of emphasis on procedural fluency and conceptual understanding (scroll to the bottom of the page and click on the left triangles under Video)

Teaching Mathematics for Robust Understanding

What Works in Math

What Works Clearinghouse Math

Researcher critiques Science of Math (create a free account to access webinar)

Resources Lambert Webinar NCTM 1_29_24

- Use of Alfieri et al. (2011) to support explicit instruction (when tested against students left alone to work on a conceptual task alone)--but the second half of the paper defines more of the study beyond efficacy of explicit instruction to show that **guided inquiry is more effective than explicit instruction.**

- Use of (Rittle-Johnson et al., 2015) as evidence that procedural and conceptual knowledge but knowledges co-develop because the activity was designed to do so. It is a the sequence of problems, designed in sequence to support conceptual development. Rittle-Johnson et al, write

" Evidence comes from studies on carefully constructed practice problems (Canobi 2009; McNeil et al. 2011, 2012, 2014). For example, elementary-school children solved packets of problems for 10 min on nine occasions during their school mathematics lessons (Canobi 2009). The problems were arithmetic problems sequenced based on conceptual principles, the same arithmetic problems sequenced randomly, or nonmathematical problems (control group). Solving conceptually sequenced practice problems supported gains in procedural as well as conceptual knowledge relative to the control condition. Solving practice problems in random order supported only modest gains in procedural knowledge and did not support gains in conceptual knowledge. Thus, **improving procedural knowledge can lead to improvements in conceptual knowledge, but not all types of procedural practice support substantial improvements in either type of knowledge**."

Does Science of Math argue that any work on procedural learning is now conceptual?

Kentaro Iwasaki’s work with districts to move towards conceptual understanding on a podcast with Harvard

[Math, the Great (Potential) Equalizer | Harvard Graduate School of Education]